An Automated Analysis of Quantum Key Distribution

R. Nagarajan[1] **N. Papanikolaou**[1] G. Bowen[2]
S. Gay[3]

[1]Department of Computer Science
University of Warwick

[2]Centre for Quantum Computation
University of Cambridge

[3]Department of Computing Science
University of Glasgow

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Outline

Quantum Information Processing

- Quantum Information Processing (QIP) is the discipline dealing with **the storage, manipulation and transmission of information using quantum phenomena.**
- QIP is divided into two interrelated areas:
  - Quantum Computation
  - Quantum Information Theory
- QIP has important applications in cryptology.

Quantum Information Processing (2)

- There exist efficient **quantum algorithms**, with no classical analogue, for solving difficult computational problems.
  - **prime factoring** and **discrete logarithm** (Peter Shor)
  - unstructured database search (Lov Grover)
- The implementation of quantum algorithms requires large–scale **quantum computers.**
- Quantum computers will clearly threaten the security of popular current–day cryptosystems (e.g. RSA, ElGamal).

Quantum Information Processing (3)

- There are several known quantum techniques for usual cryptographic tasks, including **oblivious transfer**, **bit commitment** and **key distribution**.
- We will focus on **quantum key distribution (QKD)** here.

Motivation

- Practical systems for QKD are already available commercially (viz. `www.magiqtech.com`, `www.idquantique.com`).
- The unconditional security proof of QKD holds for an **ideal** implementation and relies on complex information–theoretic arguments.
- We are in favour of a more practical approach, which is at a closer level to implementation: **probabilistic model–checking**.

Background
Key Distribution

- **Key distribution** is the process of establishing a common secret
$$k \in \{0, 1\}^N$$
known as the **key**, between two users (“Alice” and “Bob”).
- Unconditionally secure key distribution in a classical (i.e. non–quantum) setting is impossible; classical key distribution is, at best, **computationally secure.**
- Strong known security result:
  - **QKD is unconditionally secure against all attacks permitted by quantum mechanics (Mayers, 1996).**

Background

Quantum Bits

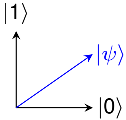
- The state of a 2–level quantum system, such as a polarised photon or a spin–$\frac{1}{2}$ particle, corresponds to a quantum bit or **qubit**.
- A qubit is a vector $|\psi\rangle$ in a 2–D complex vector space $\mathcal{H}_2$.
- The **unit length**, **orthogonal** vectors $|0\rangle$ and $|1\rangle$ form a **basis** of $\mathcal{H}_2$.
- The general state of a qubit is a linear combination

$$|\psi\rangle = \alpha \cdot |0\rangle + \beta \cdot |1\rangle, \qquad \alpha, \beta \in \mathbb{C}$$

Background

Measuring qubits (1)

- Measurements are made with respect to a given basis.
- If the qubit state $|\psi\rangle = \alpha \cdot |0\rangle + \beta \cdot |1\rangle$, is measured w.r.t $\boxplus = \{|0\rangle, |1\rangle\}$, then the state collapses into:
  - **either** $|0\rangle$, with probability $||\alpha||^2$,
  - **or** $|1\rangle$, with probability $||\beta||^2$.

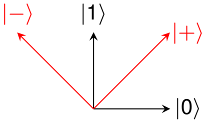


Quantum measurement is **probabilistic** and **destructive**.

Background
Measuring qubits (2)

- Consider the so–called **Hadamard basis**, which is a rotation of the computational basis by 45°. It is written $\boxtimes = \{|+\rangle, |-\rangle\}$ where:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$



- Measuring a qubit in state $|\psi\rangle = \alpha \cdot |0\rangle + \beta \cdot |1\rangle$ w.r.t. $\{|+\rangle, |-\rangle\}$ will collapse its state into:
  - **either** $|+\rangle$, with probability $||\frac{\alpha+\beta}{\sqrt{2}}||^2$,
  - **or** $|-\rangle$, with probability $||\frac{\alpha-\beta}{\sqrt{2}}||^2$.

Quantum Key Distribution (QKD)

- The security of QKD relies on the probabilistic and destructive nature of quantum measurement, as well as the **no–cloning theorem** for quantum states.
  - Quantum channels cannot be monitored without causing noticeable disturbances.
  - Quantum states cannot be cloned.
- Several protocols have been proposed for QKD:
  - **BB84 (Bennett and Brassard, 1984)**
  - B92 (Bennett, 1992)
  - E91 (Ekert, 1991)

The BB84 Protocol for QKD

1. **Alice** picks a **random bit** and a **random basis**. She encodes the bit as a qubit expressed w.r.t. the chosen basis and sends the qubit to Bob.
2. **Bob** picks a **random basis** with which to measure; he measures the qubit and stores the result.
3. **Alice** tells **Bob** which basis she used. All bits for which the wrong basis was used by Bob are discarded.

Incorrect basis:

If the incorrect basis is used for measurement, Bob obtains a random result.

Eavesdropping:

If an eavesdropper is present, there will arise cases in which both Alice and Bob used the same basis, but got a different result.

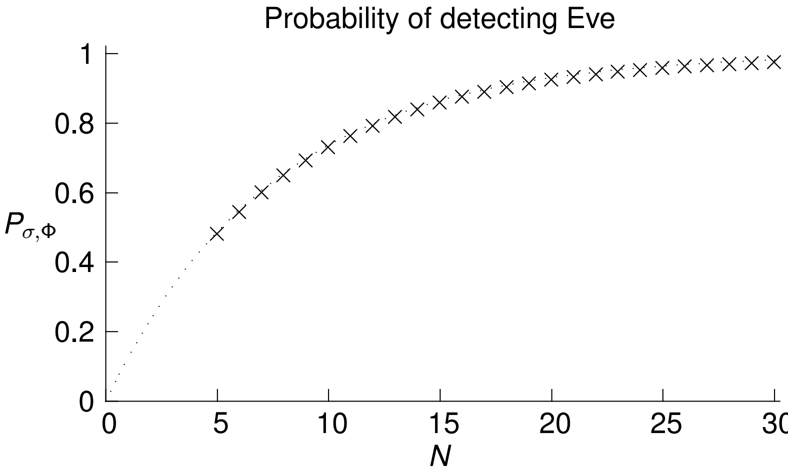
Probabilistic Model Checking

- A **probabilistic model checker** is designed to allow the verification of concurrent systems with probabilistic behaviour.
  - **PRISM (Kwiatkowska et al., 2001)**
  - ProbVerus (Clarke et al., 1999)
  - ProbUSM (Baier et al., 2005)
- For a given model σ and temporal formula ϕ, PRISM computes $\Pr(\sigma \models \phi)$.

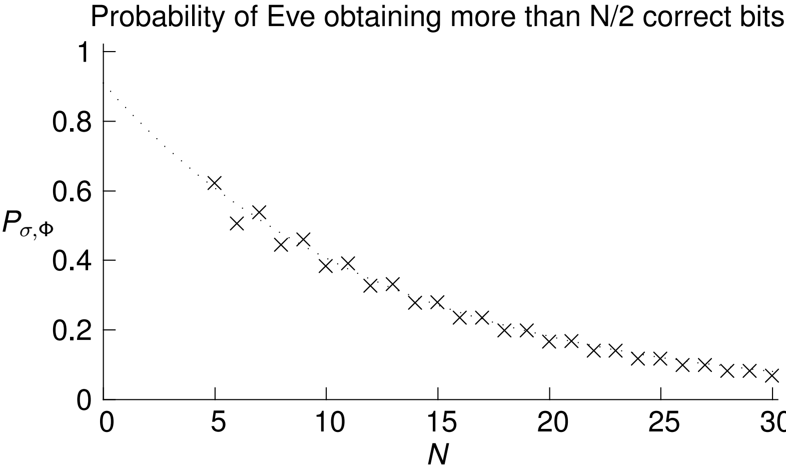
PRISM Models of BB84

- We have used PRISM to create a model of the basic BB84 protocol. With PRISM we have computed:
  - the probability P_{det} of detecting an eavesdropper when N qubits are transmitted; and
  - the probability $P_{>1/2}$ that the eavesdropper obtains more than half the originally transmitted bit values by measurement.
- The model has a single parameter, the number N of qubits transmitted by Alice to Bob over the quantum channel.
- We have computed the probabilities P_{det} and $P_{>1/2}$ for N ranging from 5 to 30.

Intercept–Resend Eavesdropping: P_{det}



Intercept–Resend Eavesdropping: $P_{>1/2}$



Current and Future Work

- Our programme is **to develop a general, high–level framework** for modelling and analysing quantum protocols using model checking.
- We are developing a **code generation tool**, PRISMGEN, which generates finite models for this purpose.
- We aim to combine our formal verification framework with a high–level specification language, in particular **CQP** (Gay and Nagarajan, 2005).

Review

- We have presented the **BB84 protocol for QKD**.
- We have conducted and discussed a **proof–of–concept analysis** of the basic BB84 protocol using probabilistic model checking.
- There is much to be done!

For Further Reading

PAPANIKOLAOU, N.
Techniques for Design and Validation of Quantum Protocols.
Master's thesis, Department of Computer Science, University of Warwick, 2005.

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Probabilistic Model–Checking of Quantum Protocols.
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Communicating Quantum Processes.
In *POPL '05: Proceedings of the 32nd ACM Symposium on Principles of Programming Languages, Long Beach, California*, January 2005.